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Analysis of Bounds for Multilinear Functions

HONG SEO RYOO[1,⋆] and NIKOLAOS V. SAHINIDIS[2]
[1]*University of Illinois at Chicago, Department of Mechanical Engineering, 842 W. Taylor Street, Chicago, Illinois 60607, USA (e-mail: hryoo@uic.edu)*
[2]*University of Illinois at Urbana-Champaign, Department of Chemical Engineering, 600 South Mathews Avenue, Urbana, Illinois 61801, USA (Corresponding author, e-mail: nikos@uiuc.edu)*

Abstract. We analyze four bounding schemes for multilinear functions and theoretically compare their tightness. We prove that one of the four schemes provides the convex envelope and that two schemes provide the concave envelope for the product of p variables over $\mathbb{R}^p_+$.

Key words: Convex envelopes, Multiplicative programs, Arithmetic intervals

1. Introduction

This paper is concerned with the bounding of multilinear functions, which are defined as:

$$\sum_{i=1}^{t} a_i \prod_{j \in J_i} y_j \tag{1}$$

Multilinear functions are the building blocks of a variety of nonconvex optimization problems. For example, they appear in bilinear, quadratic, and multiplicative programs (cf. [13]). In addition, multilinear functions arise when the Reformulation-Linearization Technique [18] is used to approximate the convex hull of general classes of mathematical programs.

Bounding multilinear functions has been an important subject in mathematical programming for over three decades now. Several linearization techniques have been developed for reformulating multilinear $0-1$ programs into mixed-integer linear programs (cf. [7, 8, 3, 4, 10, 12, 11]). However, there is relatively little work done for bounding multilinear functions of continuous variables [14, 2, 9, 6, 15, 16].

McCormick [14] gives a set of four hyperplanes for bounding y_1y_2. Al-Khayyal and Falk [2] prove that two of these hyperplanes provide the convex envelope of y_1y_2; while the other two the concave envelope. In general, convex envelopes of multilinear functions on a unit hypercube are known to be polyhedral. Rikun [15]

⋆ This research was conducted while the author was at the University of Illinois at Urbana-Champaign.

develops necessary and sufficient conditions for polyhedrality of convex envelopes and illustrates how these conditions may be used in constructing convex envelopes. Rikun [15] also provides an analytic formula defining some faces of the convex envelope of a multilinear function and gives an explicit formula for the convex envelope of the function

$$\sum_{\substack{i=1\\ j\neq i}}^{p} y_i y_j$$

over the unit hypercube. Also over the unit hypercube, Sherali [16] develops explicit formulae for the convex envelopes for the multilinear functions with coefficients that are all $+1$ or -1 via examining the convex hull representations of these functions obtained by applying the Reformulation–Linearization Technique [1, 17].

One can also bound (1) via bounds for monomial functions, which are defined as:

$$\varphi^p(y) := \prod_{j=1}^{p} y_j \tag{2}$$

The most common approach to bound the monomial over the unit hypercube is to use

$$\max\left\{\sum_{j=1}^{p} y_j - p + 1, 0\right\} \tag{3}$$

as the underestimating function and

$$\min\left\{y_j : j = 1, \ldots, p\right\} \tag{4}$$

as the overestimating function. Crama [6] proves that (3) and (4), provide, respectively, the convex and concave envelope for the monomial function in (2). In addition, [6] analyzes situations where this bounding technique leads to the convex and concave envelopes for multilinear functions over a unit hypercube $[0, 1]^p$.

It is well understood [6, 15] that bounding multilinear functions via (3)–(4) often leads to a poor approximation and, in addition, may require more hyperplanes than the convex envelope for (1). Unfortunately, finding the convex or concave envelope of a multilinear function on a unit hypercube is a $\mathcal{NP}-$hard problem [5]. Furthermore, most current results for envelopes of (1), as well as (3)-(4), are valid only over the unit hypercube. An affine transformation of variables is thus required in order to employ these results in a more general setting. Consequently, the use of the above-mentioned bounding schemes becomes somewhat problematic in the context of branching in branch-and-bound algorithms. This motivates the further study of bounds for (2) for $y \in \mathbb{R}^p$.

Hamed [9] develops three bounding schemes for (2) when $y \in \mathbb{R}^3$: the arithmetic interval method, the logarithmic transformation method, and the exponent transformation method. The present paper considers these three bounding schemes when extended to the product of arbitrarily many variables. In addition, we develop a fourth bounding scheme, which is a variant of the interval method. Our main result is an analytical comparison of the bounds obtained by these four bounding schemes.

The remainder of this paper is organized as follows. In Section 2, we derive four bounding schemes for (2). Three of these schemes apply for $y \in \mathbb{R}^p$ and one scheme requires strictly positive variables. Under the assumption that $y \in \mathbb{R}^p_+$, Section 3 presents theoretical comparisons of the tightness of the bounds that the bounding schemes provide. We provide results on lower and upper bounding functions – it is the sign of a_i in (1) that determines which of these two bounding functions of (2) must be used in bounding (1). The results include new proofs for the convex envelopes in (3) – (4); earlier proofs were provided in [6, 15, 16]. We further prove that one of the four lower bounding schemes provides the convex envelope and two provide the concave envelope of (2). Finally, conclusions are provided in Section 4.

2. Bounding Schemes

This section develops and compares four lower bounding schemes for monomial functions in (2). These bounds are based on arithmetic intervals, recursive application of arithmetic intervals, logarithmic transformation, and exponent transformation. We denote the four lower bounding schemes by `AI`, `rAI`, `Loga`, and `Expo`, respectively. We adopt the following notation for any function f:

$\overline{f}$: a concave upper bounding function of f
$\underline{f}$: a convex lower bounding function of f
$\overline{f}_{\bullet}$: $\overline{f}$ of f constructed by Scheme $\bullet$
$\underline{f}_{\bullet}$: $\underline{f}$ of f constructed by Scheme $\bullet$
$conc_f$: the concave envelope of f
$conv_f$: the convex envelope of f

2.1. ARITHMETIC INTERVALS (AI)

Let $y_j \in [y_j^L, y_j^U]$, $j = 1, \ldots, p$. For $p = 2$,

$$(y_1^U - y_1)(y_2^U - y_2) \geqslant 0$$

and

$$(y_1 - y_1^L)(y_2 - y_2^L) \geqslant 0$$

imply:

$$y_1y_2 \geqslant \max\left\{\begin{array}{c} y_1y_2^U + y_1^Uy_2 - y_1^Uy_2^U \\ y_1y_2^L + y_1^Ly_2 - y_1^Ly_2^L \end{array}\right\} = \underline{\varphi}^2_{\mathtt{AI}}(y)$$

Similarly,

$$(y_1^U - y_1)(y_2 - y_2^L) \geqslant 0$$

and

$$(y_1 - y_1^L)(y_2^U - y_2) \geqslant 0$$

imply:

$$y_1y_2 \leqslant \min\left\{\begin{array}{c} y_1y_2^L + y_1^Uy_2 - y_1^Uy_2^L \\ y_1y_2^U + y_1^Ly_2 - y_1^Ly_2^U \end{array}\right\} = \overline{\varphi}^2_{\mathtt{AI}}(y)$$

It is well-known that $\underline{\varphi}^2_{\mathtt{AI}}$ and $\overline{\varphi}^2_{\mathtt{AI}}$ are the convex and the concave envelope of y_1y_2, respectively [2]. In general, AI first generates valid underestimators of (2) by properly multiplying the variable bounds inequalities. Each of the nonlinear terms in each valid underestimator is then lower bounded, and $\underline{\varphi}^p_{\mathtt{AI}}$ is finally constructed by taking the maximum of all linear lower bounding functions of each and every valid underestimator of φ^p. For further illustration, consider $p = 3$. From

$$(y_1 - y_1^L)(y_2 - y_2^L)(y_3 - y_3^L) \geqslant 0$$

$$(y_1^U - y_1)(y_2^U - y_2)(y_3 - y_3^L) \geqslant 0$$

$$(y_1^U - y_1)(y_2 - y_2^L)(y_3^U - y_3) \geqslant 0$$

$$(y_1 - y_1^L)(y_2^U - y_2)(y_3^U - y_3) \geqslant 0$$

we obtain:

$$y_1y_2y_3 \geqslant \max\left\{\begin{array}{c} y_1y_2y_3^L + y_1y_2^Ly_3 + y_1^Ly_2y_3 \\ -y_1y_2^Ly_3^L - y_1^Ly_2y_3^L - y_1^Ly_2^Ly_3 + y_1^Ly_2^Ly_3^L \\ y_1y_2y_3^L + y_1y_2^Uy_3 + y_1^Uy_2y_3 \\ -y_1y_2^Uy_3^L - y_1^Uy_2y_3^L - y_1^Uy_2^Uy_3 + y_1^Uy_2^Uy_3^L \\ y_1y_2y_3^U + y_1y_2^Ly_3 + y_1^Uy_2y_3 \\ -y_1y_2^Ly_3^U - y_1^Uy_2y_3^U - y_1^Uy_2^Ly_3 + y_1^Uy_2^Ly_3^U \\ y_1y_2y_3^U + y_1y_2^Uy_3 + y_1^Ly_2y_3 \\ -y_1y_2^Uy_3^U - y_1^Ly_2y_3^U - y_1^Ly_2^Uy_3 + y_1^Ly_2^Uy_3^U \end{array}\right\} \quad (5)$$

Each bilinear term in (5) is then lower bounded by the maximum of two linear functions, and the resulting $\underline{\varphi}^3_{\mathtt{AI}}$ is the maximum of the $32(= 4\cdot 2^3)$ linear functions. The construct of $\overline{\varphi}^3_{\mathtt{AI}}$ is similar.

THEOREM 1. *$\underline{\varphi}^p_{AI}$ $(p = 2, 3, \dots)$ is the maximum of $\prod_{k=2}^{p-1} \Xi_k^{\binom{p}{k}} \sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p}{2i}$ linear functions, where Ξ_k denotes the number of linear functions that AI generates to lower bound k-cross-product terms, $k = 2, \dots, p-1$.*

Proof. First write out the variable bounds inequalities:

$$y_j^U - y_j \geqslant 0, \quad j = 1, \dots, p \tag{6a}$$

$$y_j - y_j^L \geqslant 0, \quad j = 1, \dots, p \tag{6b}$$

To obtain a lower bounding function of (2), take an even number of factors from (6a) and multiply them by the factors from (6b) for the remaining variables. That is,

$$\prod_{j\in I}(y_j^U - y_j) \prod_{j\in P\backslash I}(y_j - y_j^L) \geqslant 0$$

where

$$P := \{1, 2, \dots, p\}$$

and

$$I := \{i : \text{ the factor involving } y_i \text{ is taken from (6a) }\}, \quad |I| \equiv \text{even}.$$

For the case $I = \emptyset$,

$$\prod_{j=1}^{p}(y_j - y_j^L) \geqslant 0$$

gives:

$$\begin{aligned}\prod_{j=1}^{p} y_j \geqslant \sum_{i=1}^{p} y_i^L \prod_{j\neq i} y_j - \sum_{i_1=1}^{p-1}\sum_{i_2>i_1} y_{i_1}^L y_{i_2}^L \prod_{j\neq i_1,i_2} y_j + \cdots \\ - (-1)^{p-1}\sum_{i=1}^{p} y_i \prod_{j\neq i} y_j^L - (-1)^p \prod_{j=1}^{p} y_j^L\end{aligned} \tag{7}$$

Above, each of $\binom{p}{2}$ bilinear terms is lower bounded by a function that is the maximum of two linear functions, each of $\binom{p}{3}$ trilinear terms is lower bounded by the maximum of 32 linear functions, and each of the $\binom{p}{k}$ k-cross-product terms, $2 \leqslant k \leqslant p-1$, is lower bounded by a function that is the maximum of Ξ_k linear functions. The resulting is a lower bounding function of (2) that is the maximum of $\prod_{k=2}^{p-1} \Xi_k^{\binom{p}{k}}$ linear functions.

Notice that the numbers of bilinear, trilinear, ..., $(p-1)$-cross-product terms in the nonlinear lower bounding function likewise constructed for the cases $|I| = 2, 4, \ldots, 2\lfloor \frac{p}{2} \rfloor$, respectively, are the same as those in (7) and that there are $\binom{p}{l}$ possible ways to form $|I| = l$ for $l = 2, 4, \ldots, 2\lfloor \frac{p}{2} \rfloor$. This gives the desired result. □

As shown in Theorem 1, construction of $\underline{\varphi}^p_{\mathtt{AI}}$ for $p \geqslant 4$ embeds computation of $-\underline{\varphi}^l_{\mathtt{AI}}$, hence $\overline{\varphi}^l_{\mathtt{AI}}$ for all $l = p-2, p-4, \ldots$. The next theorem is concerned with $\overline{\varphi}^p_{\mathtt{AI}}$:

THEOREM 2. *$\overline{\varphi}^p_{AI}$ $(p = 2, 3, \ldots)$ is the minimum of $\prod_{k=2}^{p-1} \Xi_k^{\binom{p}{k}} \sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p}{2i}$ linear functions, where Ξ_k for $k = 2, \ldots, p-1$ are as defined in Theorem 1.*

Proof. Refer to the proof of Theorem 1. We need to combine the inequalities (6a)-(6b) so as to produce

$$-\prod_{j=1}^{p} y_j \geqslant \text{underestimating function}.$$

We achieve this by requiring $|I|$ to be odd this time and following the procedure given in the proof of Theorem 1. Now apply the same reasoning as in the proof of Theorem 1. □

REMARK 1. The number of linear maximands for $\overline{\varphi}^p_{\mathtt{AI}}$ grows doubly exponentially in p. For example, $\overline{\varphi}^4_{\mathtt{AI}}$ is the maximum of 536,870,912 linear functions. This implies that `AI` may be practically useful only in the context of a column/row generation scheme.

2.2. RECURSIVE ARITHMETIC INTERVALS (rAI)

The rationale behind the development of `rAI` is that construction of the convex and concave envelopes of the product of two variables can be accomplished easily. Factorable programming techniques [14] can then be used to utilize these bounds in building bounds for the product of p variables. The following two-step procedure summarizes `rAI` which operates on any ordering of the variables:

Step 1. Recursively replace each bilinear term in (2) with a new variable until the right hand side of (2) is replaced by a single variable. For example,

$$\underbrace{\underbrace{\underbrace{\underbrace{y_1 y_2}_{=:y_{p+1}} y_3}_{=:y_{p+2}} \cdots y_{p-1}}_{=:y_{2p-2}} y_p}_{=:y_{2p-1}}$$

Step 2. Linearly lower bound each of the $p-1$ introduced variables, *i.e.*, the bilinear terms, with the maximum of two linear functions:

$$y_j = y_{j_1} y_{j_2} \geqslant \underline{\varphi}^2_{\mathtt{rAI}}(y_{j_1}, y_{j_2}) = \max \left\{ \begin{array}{l} y_{j_1} y^U_{j_2} + y^U_{j_1} y_{j_2} - y^U_{j_1} y^U_{j_2} \\ y_{j_1} y^L_{j_2} + y^L_{j_1} y_{j_2} - y^L_{j_1} y^L_{j_2} \end{array} \right\}$$

for all $j = p+1, \ldots, 2p-1$, where j_1 and j_2 are the indices of the two original problem variables whose product is identified with variable j. By interval arithmetic, the bounds on the introduced variables are given by $y^L_j := \min \left\{ y^L_{j_1} y^L_{j_2}, y^L_{j_1} y^U_{j_2}, y^U_{j_1} y^L_{j_2}, y^U_{j_1} y^U_{j_2} \right\}$ and $y^U_j := \max \left\{ y^L_{j_1} y^L_{j_2}, y^L_{j_1} y^U_{j_2}, y^U_{j_1} y^L_{j_2}, y^U_{j_1} y^U_{j_2} \right\}$, for $j = p+1, \ldots, 2p-1$.

The following is immediate:

THEOREM 3. *$\underline{\varphi}^p_{rAI}$ is the maximum of 2^{p-1} linear functions.*

REMARK 2. (i) Note in Step 1 above that y_{p+1} can be identified with y_{1_1} and y_{1_2}, any two y_j's, $j = 1, \ldots, p$. Likewise, y_{p+2} can be identified with any two variables j such that $j \in \{j \in I\!N : 1 \leqslant j \leqslant p+1\} \setminus \{1_1, 1_2\}$. Hence, there are $\binom{p}{2} + \binom{p-1}{2} + \ldots + \binom{2}{2} = \sum_{i=2}^{p} \binom{i}{2} = \binom{p+1}{3}$ different ways to introduce bilinear relationships in Step 1.
(ii) Even though $\underline{\varphi}^p_{\mathtt{rAI}}$ is the maximum of exponentially many (2^{p-1}) linear functions, it can be represented in terms of polynomially many variables and constraints (with the addition of $p-1$ variables and $2(p-1)$ linear inequalities as shown in Step 2 above).

2.3. LOGARITHMIC TRANSFORMATION (Loga)

`Loga` is based on a basic property of the inverse functions $\exp(y)$ and $\log(y)$. Namely, for $y_j > 0$, $j = 1, \ldots, p$:

$$\prod_{j=1}^{p} y_j = \exp \left\{ \sum_{j=1}^{p} \log y_j \right\}$$

Lower bounding of $\log y_j$ above is straightforward via a secant line $f(y) = \alpha_j y_j + \beta_j$, where $\alpha_j := \frac{\log\left(y_j^U / y_j^L\right)}{y_j^U - y_j^L}$ and $\beta_j := \log y_j^U - \alpha_j y_j^U$.

2.4. EXPONENT TRANSFORMATION (Expo)

For $p = 2$ and 3 we have:

$$\begin{aligned} y_1 y_2 =& \frac{1}{8}\left\{(y_1+y_2)^2 - (y_1-y_2)^2 - (-y_1+y_2)^2 + (-y_1-y_2)^2\right\} \\ =& \frac{1}{4}\left\{(y_1+y_2)^2 - (y_1-y_2)^2\right\}; \text{ and} \\ y_1 y_2 y_3 =& \frac{1}{48}\left\{(y_1+y_2+y_3)^3 - (y_1+y_2-y_3)^3\right. \\ & - (y_1-y_2+y_3)^3 - (-y_1+y_2+y_3)^3 \\ & + (y_1-y_2-y_3)^3 + (-y_1+y_2-y_3)^3 \\ & \left. + (-y_1-y_2+y_3)^3 - (-y_1-y_2-y_3)^3\right\} \\ =& \frac{1}{24}\left\{(y_1+y_2+y_3)^3 - (y_1+y_2-y_3)^3\right. \\ & \left. - (y_1-y_2+y_3)^3 + (y_1-y_2-y_3)^3\right\} \end{aligned}$$

Expo is based on the following result:

THEOREM 4. *For $p = 2, 3, \ldots,$ the product of p variables can be separated into the sum of 2^{p-1} terms of power p in linear variables:*

$$\prod_{j=1}^{p} y_j = \frac{1}{p!2^{p-1}}\left\{\sum_{\Theta_2\in\{-1,1\}} \cdots \sum_{\Theta_p\in\{-1,1\}} \left(\prod_{j=2}^{p}\Theta_j\right)\left(y_1 + \sum_{j=2}^{p}\Theta_j y_j\right)^p\right\} \quad (8)$$

Proof. By the multinomial theorem, we have:

$$\left(y_1 + \sum_{j=2}^{p}\Theta_j y_j\right)^p = \sum_{k_1+\cdots+k_p=p} \frac{p!}{k_1!\cdots k_p!} y_1^{k_1}(\Theta_2 y_2)^{k_2}\cdots(\Theta_p y_p)^{k_p} \quad (9)$$

Use (9) to expand terms in the braces of the right hand side of (8). Group the terms by their exponents $(k_1, \ldots, k_p)$.

First, consider the group of terms corresponding to $k_1 = \cdots = k_p = 1$. Summing these terms, we obtain:

$$p!\sum_{k=1}^{p-1}\binom{p-1}{k}\prod_{j=1}^{p} y_j = p!2^{p-1}\prod_{j=1}^{p} y_j$$

Next, arbitrarily choose a combination of $(k_1, \dots, k_p)$ with $k_i > 1$ for at least one i. Then, there exists $j \neq i$ with $k_j = 0$. With respect to this choice of $(k_1, \dots, k_j(=0), \dots, k_p)$, $\left(\prod_{j=2}^p \Theta_j\right)\left(y_1 + \sum_{j=2}^p \Theta_j y_j\right)$ with $(1, \Theta_2, \dots, \Theta_j, \dots, \Theta_p)$ contributes the term

$$\Theta_j \prod_{l \neq 1, j} \Theta_l \frac{p!}{k_1! \cdots k_p!} y_1^{k_1} (\Theta_2 y_2)^{k_2} \cdots (\Theta_j y_j)^{k_j} \cdots (\Theta_p y_p)^{k_p}. \tag{10}$$

There exists, however, exactly one p-tuple of coefficients $(1, \Theta_2, \dots, -\Theta_j, \dots, \Theta_p)$ which, with respect to the same $(k_1, \dots, k_j(=0), \dots, k_p)$ combination, contributes the term

$$-\Theta_j \prod_{l \neq 1, j} \Theta_l \frac{p!}{k_1! \cdots k_p!} y_1^{k_1} (\Theta_2 y_2)^{k_2} \cdots (-\Theta_j y_j)^{k_j} \cdots (\Theta_p y_p)^{k_p}.$$

Upon summation, the above term cancels out (10). As this holds for any arbitrarily chosen combination of $(k_1, \dots, k_p)$, we have the result. □

The following two-step procedure summarizes `Expo`:

Step 1. Let c_k, $k = 1, \dots, 2^{p-1}$, denote the coefficients in front of the terms in parentheses of the right hand side of (8). Further, introduce a new variable θ_k, $k = 1, \dots, 2^{p-1}$, for each of the quantities in the parentheses of the right hand side of (8):

$$\prod_{j=1}^{p} y_j = \sum_{k=1}^{2^{p-1}} c_k \theta_k^p$$

where

$$\left.\begin{aligned} c_k &= \frac{1}{p!2^{p-1}} \prod_{j=2}^p \Theta_j \\ \theta_k &= y_1 + \sum_{j=2}^p \Theta_j y_j \end{aligned}\right\} \text{for some } \Theta_2 \in \{-1, 1\}, \dots, \Theta_p \in \{-1, 1\}$$

At this stage, the product has been separated into a sum of 2^{p-1} univariate monomials in new variables θ_k $(k = 1, \dots, 2^{p-1})$ at the expense of introducing a set of linear constraints relating the new to the original problem variables. Lower and upper bounds for θ_k, $k = 1, \dots, 2^{p-1}$, are available by interval arithmetic operations on the bounds of y_j, $j = 1, \dots, p$.

Step 2. Lower and upper bound each $c_k \theta_k^p$, $k = 1, \dots, 2^{p-1}$, using standard techniques for bounding univariate functions [14]. The sum of these lower and upper bounds provides, respectively, $\underline{\varphi}^p_{\texttt{Expo}}(y)$ and $\overline{\varphi}^p_{\texttt{Expo}}(y)$.

3. Comparison of Lower Bounding Schemes

Suppose that $y_j^L = 0$, $j = 1, \dots, p$. In this case, we may assume, without loss of generality, that all $y_j \in [0, 1]$ and compare `AI`, `rAI`, and `Expo`. We first derive the convex and the concave envelope of φ^p over $[0, 1]^p$. The results of Theorems 5 and 6 appeared earlier in [6, 15, 16] but the proofs below are new and more intuitive and provided for the reader's convenience.

THEOREM 5. *Let $y \in [0, 1]^p$, $p \geqslant 2$. Then*

$$conv_{\varphi^p}(y) = \max\left\{\sum_{j=1}^{p} y_j - (p-1), 0\right\}.$$

Proof. Let

$$f_1(y) := \sum_{j=1}^{p} y_j - (p-1); \text{ and}$$
$$f_2(y) := 0$$

We first show that $f := \max\{f_1, f_2\}$ is a lower bounding function of φ^p. Note that $-(p-1) \leqslant f_1(y) \leqslant 1$ for any $y \in [0, 1]^p$ and that, if any $y_j = 0$, then $\varphi^p \geqslant f$ because $\varphi^p = 0$. Suppose now that $y_j \neq 0$, $j = 1, \ldots, p$. In this case, define all the variables in terms of y_1, i.e.,

$$y_j := a_j y_1, a_j \in \mathbb{R}^+, \quad j = 2, \ldots, p.$$

Then

$$g(y_1) := \varphi^p(y) - f_1(y) = y_1^p \prod_{j=2}^{p} a_j - \left(1 + \sum_{j=2}^{p} a_j\right) y_1 + (p-1);$$

$$g'(y_1) = p y_1^{p-1} \prod_{j=2}^{p} a_j - \left(1 + \sum_{j=2}^{p} a_j\right);$$

and

$$g''(y_1) = p(p-1) y_1^{p-2} \prod_{j=2}^{p} a_j.$$

As $g''(y_1) > 0$ for $0 < y_1 \leqslant 1$, solving $g'(y_1) = 0$ for $y_1 = y_1^*$ we obtain the (global) minimum g^* of g:

$$g^*(y_1^*) = y_1^* \left\{\prod_{j=2}^{p} a_j \left(\frac{1 + \sum_{j=2}^{p} a_j}{p \prod_{j=2}^{p} a_j}\right) - \left(1 + \sum_{j=2}^{p} a_j\right)\right\} + p - 1$$

$$= y_1^* \left\{ -\frac{p-1}{p} \left(1 + \sum_{j=2}^{p} a_j \right) \right\} + p - 1$$

$$= p - 1 - \frac{p-1}{p} \left(1 + \sum_{j=2}^{p} a_j \right) y_1^*$$

As $\left(1 + \sum_{j=2}^{p} a_j\right) y_1^* = \sum_{j=1}^{p} y_j^* \in (0, p]$, we have:

$$g^*(y_1^*) \in p - 1 - \frac{p-1}{p} \cdot (0, p]$$
$$= [0, p-1)$$

This shows that $\varphi^p \geqslant f$. An alternate proof of this assertion is the proof of Theorem 7. Let us now show that f is the tightest lower bounding function of φ^p. Consider any $\underline{\varphi}^p$.

Case (i). Let

$$S_1 := \left\{ y \in [0,1]^p : \sum_{j=1}^{p} y_j - (p-1) \leqslant 0 \right\}$$

Note that any point in S_1 can be expressed as a convex combination of $2^p - 1$ extreme points $e(1), \ldots, e(2^p - 1)$ of S_1 and that

$$\varphi^p(e(i)) = 0, \quad i = 1, \ldots, 2^p - 1.$$

By convexity,

$$\underline{\varphi}^p(e(i)) \leqslant 0, \quad i = 1, \ldots, 2^p - 1.$$

Hence, for any $y \in S_1$, we have

$$\underline{\varphi}^p(y) = \underline{\varphi}^p \left(\sum_{i=1}^{2^p-1} \lambda_i e(i) \right) \leqslant \sum_{i=1}^{2^p-1} \lambda_i \underline{\varphi}^p(e(i)) \leqslant 0 = f_2(y)$$

where $\lambda_i \geqslant 0$ and $\sum_i \lambda_i = 1$. This shows that f_2 is the convex envelope of φ^p over S_1.

Case (ii). Let

$$S_2 := \left\{ y \in [0,1]^p : \sum_{j=1}^{p} y_j - (p-1) \geqslant 0 \right\}$$

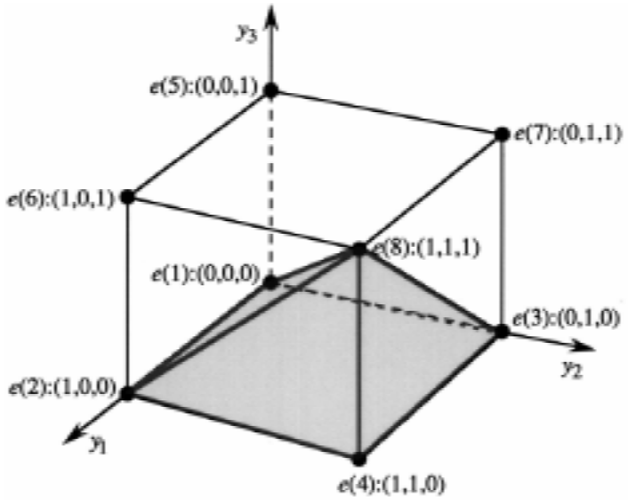

Figure 1. Plane $y_1 + y_2 + y_3 = 2$ that separates S_1 and S_2 in Theorem 5 for $p = 3$

Note that f_1 is a convex envelope of φ^p over $S_1 \cap S_2$. For any point $y \in S_2$, $y = \lambda y(p) + (1 - \lambda)e(2^p)$, where $y(p) \in S_1 \cap S_2, e(2^p) := (1, \ldots, 1)$, and $\lambda \in [0, 1]$. Consider any $\underline{\varphi}^p$. Then

$$\begin{aligned}\underline{\varphi}^p(y) &= \underline{\varphi}^p(\lambda y(p) + (1-\lambda)e(2^p)) \\ &\leqslant \lambda\underline{\varphi}^p(y(p)) + (1-\lambda)\underline{\varphi}^p(e(2^p)) \\ &\leqslant \lambda f_1(y(p)) + (1-\lambda) f_1(e(2^p)) \\ &= f_1(y).\end{aligned}$$

This proves that f_1 is the convex envelope of φ^p over S_2. As $S_1 \cup S_2 = [0, 1]^p$, cases (i) and (ii) prove the assertion of the theorem. □

THEOREM 6. *Let* $y \in [0, 1]^p$, $p \geqslant 2$. *Then*

$$conc_{\varphi^p}(y) = \min_{1 \leqslant j \leqslant p} \{y_j\}.$$

Proof. Let

$$S_i := \{y \in [0, 1]^p : y_i \leqslant y_j, \forall j \neq i\}, \quad i = 1, \ldots, p.$$

Note that each S_i is defined by $2^{p-1} + 1$ extreme points: 2^{p-1} points of the form $y_i = 0, y_j \in \{0, 1\}, \forall j \neq i$, and one with $y_j = 1 \forall_j$. Arbitrarily select S_i and denote by $e(i_1), \ldots, e(i_{2^{p-1}+1})$ the $2^{p-1} + 1$ extreme points defining S_i. Let

$$f_i(y) := y_i$$

and note that

$$f_i(e(i_k)) = \varphi^p(e(i_k)), \quad k = 1, \dots, 2^{p-1}+1.$$

Then, for any $y \in S_i$ and for $\overline{\varphi}^p$ with $\lambda_k \geqslant 0$, $\sum_k \lambda_k = 1$, we have:

$$\begin{aligned}
\overline{\varphi}^p(y) &= \overline{\varphi}^p\left(\sum_{k=1}^{2^{p-1}+1} \lambda_k e(i_k)\right) \\
&\geqslant \sum_{k=1}^{2^{p-1}+1} \lambda_k \overline{\varphi}^p(e(i_k)) \\
&\geqslant \sum_{k=1}^{2^{p-1}+1} \lambda_k \varphi^p(e(i_k)) \\
&= \sum_{k=1}^{2^{p-1}+1} \lambda_k f_i(e(i_k)) \\
&= \sum_{k=1}^{2^{p-1}+1} f_i(\lambda_k e(i_k)) \\
&= f_i\left(\sum_{k=1}^{2^{p-1}+1} \lambda_k e(i_k)\right) \\
&= f_i(y)
\end{aligned}$$

This proves that $f_i(y) = y_i$ is the concave envelope of φ^p over S_i. As $\cup_{i=1}^p S_i = [0, 1]^p$, we have the desired result. □

An illustration of Theorem 6 for $p = 3$ is provided in Figure 2.

We next study rAI in comparison to the convex and concave envelopes derived above.

THEOREM 7. *Let* $y \in [0, 1]^p$, $p \geqslant 2$. *Then*

$$\underline{\varphi}^p_{rAI}(y) = conv_{\varphi^p}(y).$$

Proof. The proof is by induction on p. For $p = 2$,

$$\left.\begin{array}{r} (y_1 - 0)(y_2 - 0) \geqslant 0 \\ (1 - y_1)(1 - y_2) \geqslant 0 \end{array}\right| \Longrightarrow \underline{\varphi}^2_{\mathtt{rAI}}(y) = \max\{y_1 + y_2 - 1, 0\}.$$

Let $p \geqslant 3$ and suppose that the assertion is true for all $l \leqslant p - 1$. Then

$$\prod_{j=1}^{p} y_j = y_p \prod_{j=1}^{p-1} y_j = y_p y_{2p-2}, \tag{11}$$

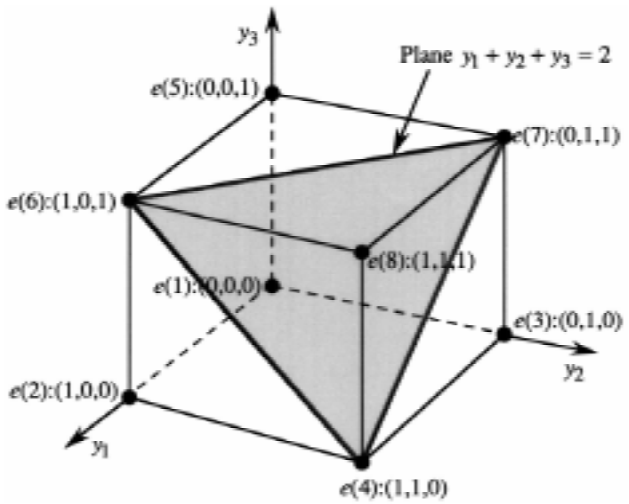

Figure 2. Illustration of Theorem 6 for $p = 3$: the part of $[0, 1]^3$ where y_3 is the concave envelope of φ^3

where

$$y_{2p-2} := \prod_{j=1}^{p-1} y_j \geqslant \underline{\varphi}_{\mathtt{rAI}}^{p-1}(y_1, \ldots, y_{p-1}) = \max\left\{\sum_{j=1}^{p-1} y_j - (p-2), 0\right\}$$

by the induction hypothesis. Continuing from (11),

$$\begin{aligned}\prod_{j=1}^{p} y_j = y_p y_{2p-2} &\geqslant \max\left\{y_p + y_{2p-2} - 1, 0\right\}\\ &\geqslant \max\left\{\sum_{j=1}^{p} y_j - (p-1), y_p - 1, 0\right\}\\ &= \max\left\{\sum_{j=1}^{p} y_j - (p-1), 0\right\}\\ &= \underline{\varphi}_{\mathtt{rAI}}^{p}(y).\end{aligned}$$

□

THEOREM 8. *Let* $y \in [0, 1]^p$, $p \geqslant 2$. *Then*

$$\overline{\varphi}_{rAI}^{p}(y) = conc_{\varphi^p}(y).$$

Proof. For $p = 2$,

$$\left.\begin{aligned}(1 - y_1)y_2 \geqslant 0\\ y_1(1 - y_2) \geqslant 0\end{aligned}\right\} \Longrightarrow y_1 y_2 \leqslant \min\{y_1, y_2\}.$$

Suppose $p \geqslant 3$. Then

$$(1-y_p)\prod_{j=1}^{p-1} y_j = (1-y_p)y_{2p-2} \geqslant 0 \Longrightarrow y_p y_{2p-2} \leqslant \min\{y_p, y_{2p-2}\}.$$

By the induction hypothesis,

$$y_{2p-2} \leqslant \overline{\varphi}^{p-1}_{\mathtt{rAI}}(y) = \min_{1\leqslant j\leqslant p-1}\{y_j\}$$

and we have

$$\overline{\varphi}^{p}_{\mathtt{rAI}}(y) \leqslant \min\left\{\overline{\varphi}^{p-1}_{\mathtt{rAI}}(y), y_p\right\} = \min_{1\leqslant j\leqslant p}\{y_j\} = conc_{\varphi^p}(y).$$

The definition of concave envelope yields the inequality in the other direction and completes the proof.

For AI, we first examine the upper bounding function that it constructs for φ^p:

THEOREM 9. *Let $y \in [0,1]^p$, $p \geqslant 2$. Then*

$$\overline{\varphi}^{p}_{AI}(y) = conc_{\varphi^p}(y).$$

Proof. $\overline{\varphi}^{2}_{\mathtt{AI}} = \overline{\varphi}^{2}_{\mathtt{rAI}}$. For any $p \geqslant 3$,

$$\left.\begin{array}{l}(1-y_1)\prod_{j=2}^{p} y_j \geqslant 0\\ \prod_{j=1}^{p-1} y_j(1-y_p) \geqslant 0\end{array}\right| \Longrightarrow \prod_{j=1}^{p} y_j \leqslant \min\left\{\prod_{j=1}^{p-1} y_j, \prod_{j=2}^{p} y_j\right\}.$$

By the induction hypothesis,

$$\overline{\varphi}^{p-1}_{\mathtt{AI}}(y_1,\ldots,y_{p-1}) = \min_{1\leqslant j\leqslant p-1}\{y_j\}$$

and

$$\overline{\varphi}^{p-1}_{\mathtt{AI}}(y_2,\ldots,y_p) = \min_{2\leqslant j\leqslant p}\{y_j\}.$$

Recall that $\overline{\varphi}^{p}_{\mathtt{AI}}(y)$ is the minimum of the convex overestimators of $\sum_{i=0}^{\lfloor \frac{p}{2}\rfloor}\binom{p}{2i}$ nonlinear upper bounding functions that AI utilizes in constructing $\overline{\varphi}^{p}_{\mathtt{AI}}$ (Refer to Theorem 2). By making use of only two of $\sum_{i=0}^{\lfloor \frac{p}{2}\rfloor}\binom{p}{2i}$ nonlinear overestimators, we obtained $\min_{1\leqslant j\leqslant p}\{y_j\}$. Hence, we have:

$$\overline{\varphi}^{p}_{\mathtt{AI}}(y) \leqslant \min_{1\leqslant j\leqslant p}\{y_j\} = conv_{\varphi^p}(y)$$

By the definition of concave envelope, we trivially have

$$\overline{\varphi}^{p}_{\mathtt{AI}}(y) \geqslant conc_{\varphi^p}(y). \qquad \square$$

Next, we derive $\underline{\varphi}^{p}_{\mathtt{AI}}$:

THEOREM 10. *Let $y \in [0,1]^p$, $p \geqslant 2$ and $P := \{1, \ldots, p\}$. Rearrange the variables so that $y_{i_1} \geqslant y_{i_2} \geqslant \ldots \geqslant y_{i_p}$ holds. Then*

$$\underline{\varphi}^p_{AI}(y) = \begin{cases} \max\left\{\sum_{j=1}^p y_j - (p-1), 0\right\}, & \text{for } p = 2, 3; \\ \max\left\{ \begin{array}{l} \sum_{j=1}^p y_j - (p-1) + \\ \sum_{l=1}^{p-3} 2^{p-3-l}\left\{\sum_{k=1}^l y_{i_k} - l\right\}, 0 \end{array} \right\}, & \text{for } p \geqslant 4. \end{cases}$$

Proof. For the base case $p = 2$, we have

$$\underline{\varphi}^p_{\mathrm{AI}}(y_i, y_j) = \max\{y_i + y_j - 1, 0\}.$$

For any $y \in [0,1]^p$, $p \geqslant 3$ and any $l \leqslant p$, define the index sets:

$$P_{p-l} := \{i_1, i_2, \ldots, i_{p-l}\}$$

and

$$\overline{P}_{p-l} := P \setminus P_{p-l} = \{i_{p-l+1}, \ldots, i_p\}.$$

Denote by $\lambda[\overline{P}_{p-l}](y)$ the nonconvex nonlinear lower bounding function of $\prod_{j=1}^p y_j$ induced by $I = \overline{P}_{p-l}$ for even values of l, where I is as defined in the proof of Theorem 1. That is, $\lambda[\overline{P}_{p-l}](y)$ is the lower bounding function of φ^p that results when the factors for the l smallest variables are taken from (6a):

$$\begin{aligned} \lambda[\overline{P}_{p-l}](y) := \prod_{j \in P_{p-l}} y_j \Bigg\{ & \sum_{j_1 \in \overline{P}_{p-l}} \sum_{\substack{j_2 \in \overline{P}_{p-l} \\ j_2 \neq j_1}} \cdots \sum_{\substack{j_{l-1} \in \overline{P}_{p-l} \\ j_{l-1} \neq j_1, \ldots, j_{l-2}}} y_{j_1} \ldots y_{j_{l-1}} \\ & - \sum_{j_1 \in \overline{P}_{p-l}} \sum_{\substack{j_2 \in \overline{P}_{p-l} \\ j_2 \neq j_1}} \cdots \sum_{\substack{j_{l-2} \in \overline{P}_{p-l} \\ j_{l-2} \neq j_1, \ldots, j_{l-3}}} y_{j_1} \ldots y_{j_{l-2}} \\ & + \cdots \\ & - \sum_{j_1 \in \overline{P}_{p-l}} \sum_{\substack{j_2 \in \overline{P}_{p-l} \\ j_2 \neq j_1}} y_{j_1} y_{j_2} \\ & + \sum_{j \in \overline{P}_{p-l}} y_j - 1 \Bigg\} \end{aligned} \tag{12}$$

Consider any $p \geqslant 3$. $I = \emptyset$ gives

$$\prod_{j=1}^p y_j \geqslant 0.$$

From $I = \overline{P}_{p-2}$, we obtain:

$$\begin{aligned}
\prod_{j=1}^{p} y_j \geqslant\ & \lambda[\overline{P}_{p-2}](y) \\
=\ & y_{i_{p-1}} \prod_{j \in P_{p-2}} y_j + y_{i_p} \prod_{j \in P_{p-2}} y_j - \prod_{j \in P_{p-2}} y_j \\
\geqslant\ & \sum_{k=1}^{p-2} y_{i_k} + y_{i_{p-1}} - (p-2) + \sum_{l=1}^{p-4} 2^{p-4-l} \left\{ \sum_{k=1}^{l} y_{i_k} - l \right\} \\
& + \sum_{k=1}^{p-2} y_{i_k} + y_{i_p} - (p-2) + \sum_{l=1}^{p-4} 2^{p-4-l} \left\{ \sum_{k=1}^{l} y_{i_k} - l \right\} - y_{i_{p-2}} \\
=\ & \sum_{j=1}^{p} y_j - (p-1) + 2 \sum_{l=1}^{p-4} 2^{p-4-l} \left\{ \sum_{k=1}^{l} y_{i_k} - l \right\} + \sum_{k=1}^{p-3} y_{i_k} - (p-3) \\
=\ & \sum_{j=1}^{p} y_j - (p-1) + \sum_{l=1}^{p-3} 2^{p-3-l} \left\{ \sum_{k=1}^{l} y_{i_k} - l \right\} \\
=:\ & \underline{\lambda}[\overline{P}_{p-2}]_{\mathtt{AI}}(y)
\end{aligned}$$

By the choice of P_{p-2}, $\underline{\lambda}[\overline{P}_{p-2}]_{\mathtt{AI}}(y)$ above is the tightest among all linear lower bounding functions of φ^p constructed by AI for the case $|I| = 2$. It remains to show that $\underline{\lambda}[\overline{P}_{p-2}]_{\mathtt{AI}}(y)$ is also tighter than the linear lower bounding functions of φ^p constructed by AI for the remaining cases, $|I| = 4, 6, \ldots, 2\lfloor \frac{p}{2} \rfloor$.

Consider any $4 \leqslant l \leqslant 2\lfloor \frac{p}{2} \rfloor$, l an even number. First note that the tightest lower bounding function of φ^p by AI for $|I| = l$ is constructed via $I = \overline{P}_{p-l}$. Hence, we will examine $\lambda[\overline{P}_{p-l}]$ and compare $\underline{\lambda}[\overline{P}_{p-l}]_{\mathtt{AI}}$ with $\underline{\lambda}[\overline{P}_{p-2}]_{\mathtt{AI}}$ above. Referring to (12), notice that there are $\binom{l}{l-1}$ positive $(p-1)$-cross-product terms, $\binom{l}{l-2}$ negative $(p-2)$-cross-product terms, $\ldots$, $\binom{l}{l-i}$ $(-1)^{i+1}$-tive $(p-l+i)$-cross-product terms, $\ldots$, l positive $(p-l+1)$-cross-product terms, and one negative $(p-l)$-cross-product term in $\lambda[\overline{P}_{p-l}]$. Rewrite $\lambda[\overline{P}_{p-l}]$ as follows:

$$\begin{aligned}
\lambda[\overline{P}_{p-l}](y) &= y_{i_1} \ldots y_{i_{p-2}} y_{i_{p-1}} + y_{i_1} \ldots y_{i_{p-2}} y_{i_p} - y_{i_1} \ldots y_{i_{p-2}} + g(y) \\
&= \lambda[\overline{P}_{p-2}](y) + g(y)
\end{aligned}$$

We will show that $\underline{g}_{\mathtt{AI}}(y)$ is never positive, hence prove $\underline{\lambda}[\overline{P}_{p-2}]_{\mathtt{AI}}(y) \geqslant \underline{\lambda}[\overline{P}_{p-l}]_{\mathtt{AI}}(y)$.

To simplify the task, let us compute $\underline{\hat{g}}_{\mathtt{AI}}$, an overestimate of $\underline{g}_{\mathtt{AI}}$ and show that $\underline{\hat{g}}_{\mathtt{AI}} \leqslant 0$. As $\underline{g}_{\mathtt{AI}} \leqslant \underline{\hat{g}}_{\mathtt{AI}} \leqslant 0$, this will establish the desired result.

Note that an overestimator of $\underline{g}_{\mathtt{AI}}$ is obtained if the piecewise linear underestimator $\max\{\sum_{j \in I_{k_i}} y_j - (k-1), 0\}$ is utilized for each positive k-cross-product term of g, where I_{k_i} denotes the index set of variables appearing in the k-cross-product term

($k = p-l+1, p-l+3, \dots, p-1$ and $i = 1, 2, \dots, \binom{l}{k-(p-l)}$($-2$ if $k = p-1$)). Further note that $\max\{\sum_{j=I_{k_i}} y_j - (k-1), 0\} \leqslant \min_{j\in I_{k_i}}\{y_j\}$.

We use $\min_{j\in I_{k_i}}\{y_j\}$ for the positive k-cross-product terms of g and obtain the following overestimator of $\underline{g}_{\mathtt{AI}}$:

$$
\begin{aligned}
\hat{\underline{g}}_{\mathtt{AI}}(y) = \;& \left\{\binom{l-1}{0}+\binom{l-1}{2}+\binom{l-1}{4}+\cdots+\binom{l-1}{l-4}+\binom{l-1}{l-2}-1\right\} y_{i_p}\\
&+\left\{\binom{l-2}{0}+\binom{l-2}{2}+\binom{l-2}{4}+\cdots+\binom{l-2}{l-4}+\binom{l-2}{l-2}-1\right\} y_{i_{p-1}}\\
&+\left\{\binom{l-3}{0}+\binom{l-3}{2}+\binom{l-3}{4}+\cdots+\binom{l-3}{l-4}\right\} y_{i_{p-2}}\\
&+\left\{\binom{l-4}{0}+\binom{l-4}{2}+\binom{l-4}{4}+\cdots+\binom{l-4}{l-4}\right\} y_{i_{p-3}}\\
&+\cdots\\
&+\left\{\binom{l-k-1}{0}+\binom{l-k-1}{2}+\cdots\right.\\
&\qquad\qquad +\binom{l-k-1}{l-k-1}\text{ (if } k \text{ is odd; or)}\\
&\qquad\qquad \left.+\binom{l-k-1}{l-k-2}\text{ (if } k \text{ is even)}\right\} y_{i_{p-k}}\\
&+\cdots\\
&+\left\{\binom{3}{0}+\binom{3}{2}\right\} y_{i_{p-l+4}}\\
&+\left\{\binom{2}{0}+\binom{2}{2}\right\} y_{i_{p-l+3}}\\
&+\binom{1}{0} y_{i_{p-l+2}}\\
&+\binom{0}{0} y_{i_{p-l+1}}\\
&-\left\{\binom{l-1}{1}+\binom{l-1}{3}+\binom{l-1}{5}+\cdots+\binom{l-1}{l-5}+\binom{l-1}{l-3}\right\} y_{i_p}\\
&-\left\{\binom{l-2}{1}+\binom{l-2}{3}+\binom{l-2}{5}+\cdots+\binom{l-2}{l-5}+\binom{l-2}{l-3}\right\} y_{i_{p-1}}\\
&-\left\{\binom{l-3}{1}+\binom{l-3}{3}+\binom{l-3}{5}+\cdots+\binom{l-3}{l-3}-1\right\} y_{i_{p-2}}\\
&-\left\{\binom{l-4}{1}+\binom{l-4}{3}+\binom{l-4}{5}+\cdots+\binom{l-4}{l-5}\right\} y_{i_{p-3}}\\
&-\cdots\\
&-\left\{\binom{l-k-1}{1}+\binom{l-k-1}{3}+\cdots\right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad + \binom{l-k-1}{l-k-2} \text{ (if } k \text{ is odd; or)} \\
&\qquad\qquad\qquad\qquad + \binom{l-k-1}{l-k-1} \text{ (if } k \text{ is even)} \Big\} y_{i_{p-k}} \\
&- \cdots \\
&- \left\{ \binom{3}{1} + \binom{3}{3} \right\} y_{i_{p-l+4}} \\
&- \binom{2}{1} y_{i_{p-l+3}} \\
&- \binom{1}{1} y_{i_{p-l+2}} \\
&- \begin{cases} 1, & \text{if } l = p; \\ y_{i_{p-l}}, & \text{otherwise} \end{cases} \\
= \;& y_{i_{p-2}} + y_{i_{p-l+1}} - y_{i_{p-1}} - \begin{cases} 1, & \text{if } l = p; \\ y_{i_{p-l}}, & \text{otherwise} \end{cases} \\
\leqslant \;& 0.
\end{aligned}
$$

This completes the proof. □

We have the following results:

THEOREM 11. *Let* $y \in [0,1]^p$, $p \geqslant 2$. *Then*

(i) $\underline{\varphi}^p_{rAI}(y) \geqslant \underline{\varphi}^p_{AI}(y)$

(ii) $\underline{\varphi}^p_{rAI}(y) \geqslant \underline{\varphi}^p_{Expo}(y)$.

Proof. Immediate from the definition of convex envelope and the fact that $\underline{\varphi}^p_{\mathtt{rAI}}$ is the convex envelope of φ^p and that $\sum_{k=1}^{l} y_{i_k} \leqslant l$. □

Consider $y \in [1,2]^p$ and compare the four bounding schemes.

EXAMPLE 1. For $p = 3$ and for $y \in [1,2]^3$, we have

$$
\underline{\varphi}^3_{\mathtt{AI}} = \max \left\{ \begin{array}{c}
2f_{12}(y_1, y_2) + 2f_{13}(y_1, y_3) + f_{23}(y_2, y_3) \\
-4y_1 - 2y_2 - 2y_3 + 4 \\
2f_{12}(y_1, y_2) + f_{13}(y_1, y_3) + 2f_{23}(y_2, y_3) \\
-2y_1 - 4y_2 - 2y_3 + 4 \\
f_{12}(y_1, y_2) + 2f_{13}(y_1, y_3) + 2f_{23}(y_2, y_3) \\
-2y_1 - 2y_2 - 4y_3 + 4 \\
f_{12}(y_1, y_2) + f_{13}(y_1, y_3) + f_{23}(y_2, y_3) \\
-y_1 - y_2 - y_3 + 1
\end{array} \right\};
$$

Table 1. Values of lower bounding functions at feasible points

y	$\underline{\varphi}^3_{\mathrm{AI}}$	$\underline{\varphi}^3_{\mathrm{rAI}}$	$\underline{\varphi}^3_{\mathrm{Loga}}$	$\underline{\varphi}^3_{\mathrm{Expo}}$	$\varphi^3 = y_1y_2y_3$
(1,1,1.1)	1.1	1.1	1.07	0.08	1.10
(1.5,1.5,1.5)	2.5	2.5	2.83	2.11	3.38
(1.5,1.5,1.6)	2.8	2.6	3.03	2.33	3.6
(2,2,2)	8	8	8	6.75	8

$$\underline{\varphi}^3_{\mathrm{rAI}} = \max \left\{ \begin{array}{c} 2f_{12}(y_1, y_2) + 4y_3 - 8 \\ f_{12}(y_1, y_2) + y_3 - 1 \end{array} \right\};$$

$$\underline{\varphi}^3_{\mathrm{Loga}} = e^{(y_1+y_2+y_3-3)}; \text{ and}$$

$$\underline{\varphi}^3_{\mathrm{Expo}} = \frac{1}{24}\left\{(y_1+y_2+y_3)^3 - 9(y_1+y_2+y_3)\right\},$$

where

$$f_{ij}(y_i, y_j) = \max \left\{ \begin{array}{c} 2y_i + 2y_j - 4 \\ y_i + y_j - 1 \end{array} \right\}.$$

Table 1 records the values of these lower bounding functions and $\varphi^3 = y_1y_2y_3$ at four feasible points of $[1, 2]^3$.

Example 1 (Table 1) proves the following dominance relationships among the four lower bounding schemes:

THEOREM 12. *Let* $y_j \in [1, 2]$, $j = 1, 2, 3$. *Then*

(i) *Neither* $\underline{\varphi}^3_{AI}(y)$ *nor* $\underline{\varphi}^3_{Loga}(y)$ *globally dominates the other;*
(ii) *Neither* $\underline{\varphi}^3_{rAI}(y)$ *nor* $\underline{\varphi}^3_{Loga}(y)$ *globally dominates the other.*

We conclude this section with three conjectures:

CONJECTURE 1. *Let* $y_j \in (0, +\infty)$, $j = 1, \ldots, p$. *Then*

(i) *Neither* $\underline{\varphi}^p_{AI}(y)$ *nor* $\underline{\varphi}^p_{Loga}(y)$ *globally dominates the other;*
(ii) *Neither* $\underline{\varphi}^p_{rAI}(y)$ *nor* $\underline{\varphi}^p_{Loga}(y)$ *globally dominates the other.*

The above conjecture is based upon the observation that $\underline{\varphi}^p_{\mathrm{AI}}$ and $\underline{\varphi}^p_{\mathrm{rAI}}$ are tighter near the boundaries of the box while $\underline{\varphi}^p_{\mathrm{Loga}}$ is tighter in the interior around the middle of the box.

CONJECTURE 2. *For* $y_j \in (0, +\infty)$, $j = 1, \dots, p$,

$$\underline{\varphi}^p_{Loga}(y) \geqslant \underline{\varphi}^p_{Expo}(y).$$

For $y_j \in (-\infty, +\infty)$, $j = 1, \dots, p$,

(i) $\underline{\varphi}^p_{AI}(y) \geqslant \underline{\varphi}^p_{Expo}(y)$;

(ii) $\underline{\varphi}^p_{rAI}(y) \geqslant \underline{\varphi}^p_{Expo}(y)$.

This conjecture is based on the observation that $\underline{\varphi}^p_{\mathtt{Expo}}$ is never exact to φ^p at any feasible point $y \in [y_j^L, y_j^U]^p$ for $p \geqslant 2$ and the difference between φ^p and $\underline{\varphi}^p_{\mathtt{Expo}}$ drastically increases, even at the extreme points of the hypercube, as p increases. Conjecture 2 was computationally verified for $p = 3, \dots, 9$ and $y \in [0.00001, 1]$.

CONJECTURE 3. *Let* $y_j \in (0, +\infty)$, $j = 1, \dots, p$. *Then*

$$\underline{\varphi}^p_{AI}(y) \geqslant \underline{\varphi}^p_{rAI}(y).$$

4. Conclusions

In this paper, we compared the tightness of four bounding schemes that can be incorporated into algorithms for solving programs involving multilinear functions. We proved that recursive arithmetic intervals provide the convex envelope while arithmetic intervals and recursive arithmetic intervals provide the concave envelope for monomial functions over the positive orthant.

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References

1. Adams, W.P. and Sherali. H.D. (1990), Linearization Strategies for A Class of Zero-One Mixed Integer Programming Problems. *Operations Research* 38(2): 217–226.
2. Al-Khayyal, F. and Falk, J.E. (1983), Jointly Constrained Biconvex Programming. *Mathematics of Operations Research* 8(2): 273–286.
3. Balas, E. and Mazzola, J.B. (1984), Nonlinear 0 – 1 Programming: I. Linearization Techniques. *Mathematical Programming* 30: 1–21.

4. Balas, E. and Mazzola. J.B. (1984), Nonlinear $0-1$ Programming: II. Dominance Relations and Algorithms. *Mathematical Programming* 30: 22–45.
5. Crama, Y. (1989), Recognition Problems for Polynomials in $0-1$ Variables. *Mathematical Programming* 44: 139–155.
6. Crama, Y. (1993), Concave Extensions for Nonlinear $0-1$ Maximization Problems. *Mathematical Programming* 61: 53–60.
7. Glover, F. and Woolsey, E. (1973), Further Reduction of Zero-One Polynomial Programs to Zero-One Linear Programs. *Operations Research* 21(1).
8. Glover, F. and Woolsey, E. (1974), Further Reduction of the $0-1$ Polynomial Program Problem to a $0-1$ Linear Program. *Operations Research* 22: 180–182.
9. Hamed, A.S.E. (1991), *Calculation of Bounds on Variables and Underestimating Convex Functions for Nonconvex Functions*. PhD thesis, The George Washington University.
10. Hammer, P.L. Hansen, P. and Simeone, B. (1984), Roof Duality, Complementation and Persistency in Quadratic $0-1$ Optimization. *Mathematical Programming* 28: 121–155.
11. P. Hansen, B. Jaumard, and M. Vincent. Constrained Nonlinear $0-1$ Programming. *ORSA Journal on Computing*, 5(2):97–119, 1993.
12. P. Hansen, S.H. Lu, and B. Simeone. On the Equivalence of Paved Duality and Standard Linearization in Nonlinear $0-1$ Optimization. *Discrete Applied Mathematics*, 29:187–193, 1990.
13. R. Horst and H. Tuy. *Global Optimization: Deterministic Approaches*. Springer-Verlag, Berlin, Heidelberg, Germany, 3rd edition, 1996.
14. McCormick, G.P. (1976), Computability of Global Solutions to Factorable Nonconvex Programs: Part I — Convex Underestimating Problems. *Mathematical Programming* 10: 147–175.
15. Rikun, A. (1997), A Convex Envelope Formula for Multilinear Functions. *Journal of Global Optimization* 10: 425–437.
16. Sherali, H.D. (1997), Convex Envelopes of Multilinear Functions over a Unit Hypercube and over Special Discrete Sets. *ACTA Mathematica Vietnamica* 22(1): 245–270.
17. Sherali, H.D. and Adams, W.P. (1990), A Hierarchy of Relaxations Between the Continuous and Convex Hull Representations for Zero-One Programming Problems. *SIAM Journal of Discrete Mathematics* 3(3): 411–430.
18. Sherali, H.D. and Adams, W.P. (1999), *Reformulation-Linearization Techniques in Discrete and Continuous Optimization*. Nonconvex Optimization and Its Applications. Kluwer Academic Publishers, Dordrecht/Boston/London.